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LETTER TO THE EDITOR

Super Yangian double $DY(gl(1|1))$ and its Gauss decomposition

Jin-fang Cai†, Guo-xing Ju†, Ke Wu† and Shi-kun Wang‡

† Institute of Theoretical Physics, Academia Sinica, Beijing, 100080, People's Republic of China
‡ CCAST (World Laboratory), PO Box 3730, Beijing, 100080, People's Republic of China and Institute of Applied Mathematics, Academia Sinica, Beijing, 100080, People's Republic of China

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Abstract. We extend the Yangian double to the super (or graded) case and give its Drinfel'd generators a realization by Gauss decomposition.

Quantum algebras (in the general meaning, this includes Yangians and quantum affine algebras etc) are new algebraic structures which were discovered about ten years ago [1–5]. They play an important role in the study of soluble statistical models and quantum field theory. Quantum universal enveloping algebras of simple Lie algebras, which are related with some simple solutions (without spectral parameter) of the Yang–Baxter equation, have been extensively and deeply studied in the past few years. Quantum affine (or Kac–Moody) algebras and Yangians are related respectively to trigonometric and rational solutions of the Yang–Baxter equation. They have three realizations in the literature: Chevalley generators, $T^{\pm}$-matrix and Drinfel'd generators. The first realization was proposed independently by Drinfel'd and Jimbo [4], the second realization has a direct meaning in the quantum inverse scattering method [5] and it is convenient to introduce a central extension using this realization [6]. The isomorphism of the $T^{\pm}$-matrix and Drinfel'd generator realizations of the quantum affine algebra was established through Gauss decomposition by Ding and Frenkel [7]. The Yangian can be viewed as a deformation of a universal enveloping algebra of only half of the corresponding loop algebra, while a Yangian double is a deformation of the complete loop algebra. Yangian doubles (and with central extension) have been studied by Bernard, Khoroshkin, Iohara and coworkers [8–14]. The properties of super Yangians and their representations have been studied by some authors [15, 16]. In this paper, we extend the Yangian double to the super case and give its Drinfel'd generators a realization through Gauss decomposition.

Given a two-dimensional graded vector space V and letting its second basis be odd, then the super (graded) Yang–Baxter equation [17–19] takes the form

$$\eta_{12}R_{12}(u)\eta_{13}R_{13}(u+v)\eta_{23}R_{23}(v) = \eta_{23}R_{23}(v)\eta_{13}R_{13}(u+v)\eta_{12}R_{12}(u) \qquad (1)$$

where $R(u) \in \mathrm{End}(V \otimes V)$ must obey the weight conservation condition $R_{ij,kl} \neq 0$ only when $i + j = k + l$. $\eta_{ik,jl} = (-1)^{(i-1)(k-1)}\delta_{ij}\delta_{lk}$. In dealing with the tensor product in the

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graded case, we must use the form $(A \otimes B)(C \otimes D) = (-1)^{P(B)P(C)} AC \otimes BD$. The super Yang–Baxter equation can also be written in components as follows:

$$R_{ib,pr}(u) R_{pa,js}(u+v) R_{rs,dc}(v) (-1)^{(r-1)(s+a)} \\ = (-1)^{(e-1)(f+c)} R_{ba,ef}(v) R_{if,kc}(u+v) R_{ke,jd}(u). \tag{2}$$

It is easy to prove that $R(u) = uI + \hbar\mathcal{P}$ satisfies the above super Yang–Baxter equation, where $\mathcal{P}_{ik,jl} = (P\eta)_{ik,jl} = (-1)^{(i-1)(k-1)}\delta_{il}\delta_{jk}$ is the permutation operator in the graded case.

Similar to $DY(gl_2)$ [13], the super Yangian double $DY(gl(1|1))$ is a Hopf algebra generated by $\{t^k_{ij} | 1 \leqslant i, j \leqslant 2, k \in \mathbb{Z}\}$ which obeys the following relations:

$$R(u-v) T^{\pm}_1(u) \eta T^{\pm}_2(v) \eta = \eta T^{\pm}_2(v) \eta T^{\pm}_1(u) R(u-v) \\ R(u-v) T^{+}_1(u) \eta T^{-}_2(v) \eta = \eta T^{-}_2(v) \eta T^{+}_1(u) R(u-v). \tag{3}$$

Here we use standard notation, $T^{\pm}_1(u) = T^{\pm}(u) \otimes \mathbf{1}$, $T^{\pm}_2(u) = \mathbf{1} \otimes T^{\pm}(u)$ and $T^{\pm}(u) = (t^{\pm}_{ij}(u))_{1\leqslant i,j\leqslant 2}$, $t^{\pm}_{ij}(u)$ are generator functions of t^k_{ij}:

$$t^{+}_{ij}(u) = \delta_{ij} - \hbar \sum_{k\geqslant 0} t^k_{ij} u^{-k-1} \qquad t^{-}_{ij}(u) = \delta_{ij} + \hbar \sum_{k<0} t^k_{ij} u^{-k-1}. \tag{4}$$

The Hopf pairing relation between $T^+(u)$ and $T^-(v)$ is defined as

$$\langle T^{+}_1(u), T^{-}_2(v) \rangle = R(u-v). \tag{5}$$

From relations (3), we can get the (anti-)commutation relations among $\{t^k_{ij}(u)\}$

$$(u-v)[t^{\sigma}_{ij}(u), t^{\rho}_{kl}(v)\} + (-1)^{ij+jk+ki+1}\hbar (t^{\sigma}_{kj}(u) t^{\rho}_{il}(v) - t^{\rho}_{kj}(v) t^{\sigma}_{il}(u)) = 0 \tag{6}$$

where (σ, ρ) takes the values $(+,+)$, $(-,-)$ and $(+,-)$. The parity of $t^{\pm}_{ij}$ is equal to $(i-1)(j-1)$ which means that $(t^{\pm}_{11}(u), t^{\pm}_{22}(u))$ are even and $(t^{\pm}_{12}(u)\ , t^{\pm}_{21}(u))$ are odd. We denote $[\ ,\ \}$ as the super commutator which is an anti-commutator only when each of the elements in it is odd (or Fermionic). The following are some special examples of the above relation:

$$(u-v)[t^{\sigma}_{11}(u), t^{\rho}_{12}(v)] + \hbar (t^{\sigma}_{11}(u) t^{\rho}_{12}(v) - t^{\rho}_{11}(v) t^{\sigma}_{12}(u)) = 0 \\ (u-v)[t^{\sigma}_{22}(u), t^{\rho}_{12}(v)] - \hbar (t^{\sigma}_{12}(u) t^{\rho}_{22}(v) - t^{\rho}_{12}(v) t^{\sigma}_{22}(u)) = 0 \\ (u-v)\{t^{\sigma}_{12}(u), t^{\rho}_{21}(v)\} - \hbar (t^{\sigma}_{22}(u) t^{\rho}_{11}(v) - t^{\rho}_{22}(v) t^{\sigma}_{11}(u)) = 0. \tag{7}$$

The Hopf structure of $DY(gl(1|1))$ is defined as follows:

$$\Delta t^{\pm}_{ij}(u) = \sum_{k=1,2} t^{\pm}_{kj}(u) \otimes t^{\pm}_{ik}(u) (-1)^{(k+i)(k+j)} \\ \epsilon(t^{\pm}_{ij}(u)) = \delta_{ij} \qquad S(^{st}T^{\pm}(u)) = [^{st}T^{\pm}(u)]^{-1} \tag{8}$$

where $[^{st}T^{\pm}(u)]_{ij} = (-1)^{i+j} t^{\pm}_{ji}(u)$.

We can obtain the Drinfel'd generator realization of $DY(gl(1|1))$ by Gauss decomposition in the same way which has been used in $DY(gl_2)$. Introducing a transformation for generator functions as follows,

$$E^{\pm}(u) = \frac{1}{\hbar} t^{\pm}_{11}\left(u + \frac{\hbar}{2}\right)^{-1} t^{\pm}_{12}\left(u + \frac{\hbar}{2}\right) \qquad F^{\pm}(u) = \frac{1}{\hbar} t^{\pm}_{21}\left(u + \frac{\hbar}{2}\right) t^{\pm}_{11}\left(u + \frac{\hbar}{2}\right)^{-1} \\ k^{\pm}_1(u) = t^{\pm}_{11}(u) \qquad k^{\pm}_2(u) = t^{\pm}_{22}(u) - t^{\pm}_{21}(u) t^{\pm}_{11}(u)^{-1} t^{\pm}_{12}(u) \tag{9}$$

this means that

$$T^{\pm}(u) = \begin{pmatrix} 1 & \\ \overline{h}F^{\pm}(u-\frac{\overline{h}}{2}) & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(u) & \\ & k_2^{\pm}(u) \end{pmatrix} \begin{pmatrix} 1 & \overline{h}E^{\pm}\left(u-\frac{\overline{h}}{2}\right) \\ & 1 \end{pmatrix}$$
$$= \begin{pmatrix} k_1^{\pm}(u) & \overline{h}k_1^{\pm}(u)E^{\pm}(u-\frac{\overline{h}}{2}) \\ \overline{h}F^{\pm}(u-\frac{\overline{h}}{2})k_1^{\pm}(u) & k_2^{\pm}(u)+\overline{h}^2F^{\pm}(u-\frac{\overline{h}}{2})k_1^{\pm}(u)E^{\pm}(u-\frac{\overline{h}}{2}) \end{pmatrix} \tag{10}$$

and we let

$$E(u) = E^{+}(u) - E^{-}(u) \qquad F(u) = F^{+}(u) - F^{-}(u)$$
$$H^{\pm}(u) = k_2^{\pm}\left(u+\frac{\overline{h}}{2}\right)k_1^{\pm}\left(u+\frac{\overline{h}}{2}\right)^{-1} \qquad K^{\pm}(u) = k_2^{\pm}\left(u+\frac{\overline{h}}{2}\right)k_1^{\pm}\left(u-\frac{\overline{h}}{2}\right). \tag{11}$$

By the similar calculation process which has been used in the quantum affine algebra [7], we can obtain the fact that these new generator functions $E(u)$, $F(u)$, $H^{\pm}(u)$ and $K^{\pm}(u)$ satisfy the following commutation relations:

$$\begin{aligned}
&[H^{\sigma}(u), H^{\rho}(v)] = [H^{\sigma}(u), K^{\rho}(v)], \qquad \forall\sigma,\rho = +,- \\
&[K^{\sigma}(u), K^{\rho}(v)] = 0 \\
&[H^{\pm}(u), E(v)] = [H^{\pm}(u), F(v)] = 0 \\
&E(v)K^{\pm}(u) = \frac{u-v+\overline{h}}{u-v-\overline{h}}K^{\pm}(u)E(v) \\
&F(v)K^{\pm}(u) = \frac{u-v-\overline{h}}{u-v+\overline{h}}K^{\pm}(u)F(v) \\
&\{E(u), E(v)\} = \{F(u), F(v)\} = 0 \\
&\{E(u), F(v)\} = \frac{1}{\overline{h}}\delta(u-v)(H^{-}(v) - H^{+}(u)).
\end{aligned} \tag{12}$$

The coproduct and counit structure for $E(u)$, $F(u)$, $H^{\pm}(u)$ and $K^{\pm}(u)$ can be calculated out as follows:

$$\begin{aligned}
&\Delta E^{\pm}(u) = E^{\pm}(u)\otimes 1 + K^{\pm}(u)\otimes E^{\pm}(u) \\
&\Delta F^{\pm}(u) = 1\otimes F^{\pm}(u) + F^{\pm}(u)\otimes K^{\pm}(u) \\
&\Delta K^{\pm}(u) = K^{\pm}(u)\otimes K^{\pm}(u) - F^{\pm}(u-\overline{h})K^{\pm}(u)\otimes K^{\pm}(u)E^{\pm}(u-\overline{h}) \\
&\Delta H^{\pm}(u) = H^{\pm}(u)\otimes H^{\pm}(u) \\
&\epsilon(K^{\pm}(u)) = \epsilon(H^{\pm}(u)) = 1 \\
&\epsilon(E^{\pm}(u)) = \epsilon(F^{\pm}(u)) = 0.
\end{aligned} \tag{13}$$

Expanding $H^{\pm}(u)$, $K^{\pm}(u)$, $E(u)$ and $F(u)$ in Fourier series

$$\begin{aligned}
&E(u) = \sum_{k\in Z} e_k u^{-k-1} \qquad F(u) = \sum_{k\in Z} f_k u^{-k-1} \\
&H^{+}(u) = 1 + \overline{h}\sum_{k\geqslant 0} h_k u^{-k-1} \qquad H^{-}(u) = 1 - \overline{h}\sum_{k<0} h_k u^{-k-1} \\
&K^{+}(u) = 1 + \overline{h}\sum_{l\geqslant 0} k_l u^{-l-1} \qquad K^{-}(u) = 1 - \overline{h}\sum_{l<0} k_l u^{-l-1}
\end{aligned} \tag{14}$$

then the commutation relations among h_l, k_l, e_l and f_l give Drinfel'ds generator realization of $DY(gl(1|1))$:

$$\begin{aligned}
&[h_k, h_l] = [k_k, k_l] = [h_k, k_l] = 0 \\
&[h_k, e_l] = [h_k, f_l] = 0 \\
&[k_0, e_l] = -2e_l, [k_0, f_l] = 2f_l \\
&[k_{k+1}, e_l] - [k_k, e_{l+1}] + \overline{h}(k_k e_l + e_l k_k) = 0
\end{aligned}$$

$$
\begin{aligned}
&[k_{k+1}, f_l] - [k_k, f_{l+1}] - \hbar(k_k f_l + f_l k_k) = 0 \\
&\{e_k, f_l\} = -2h_{k+l} \\
&\{e_k, e_l\} = \{f_k, f_l\} = 0.
\end{aligned}
\tag{15}
$$

It is natural that we may also be able to discuss the central extension of $DY(gl(1|1)$ and that the Gauss decomposition can be applied to the quantum affine superalgebra to obtain the Ding–Frenkel map of its two realizations. As to its physical application, Yangian symmetry has been discovered in an integrable quantum field [8, 9], spin chains with long-range interactions (including Calogero–Sutherland models) [20, 21], the Hubbard model [22], a conformal field [23, 24] and so on. The super Yangian has also been applied in colour Calogero–Sutherland models [25]. It may be that the symmetries in these systems are the corresponding (super) Yangian doubles as in [8].

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